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Dynamical Structures for k-Vector Fields

Geoffrey Martin[1]

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A new class of dynamical structures that generalize electrodynamics is presented. In this construction the 1-jets of solutions are represented by a class of k-vector fields that extend the notion of a Poisson structure to multivectors of degree greater than two. These objects function as tangent vectors to solutions. Although the dynamical equations are systems of partial differential equations, the formalism is very similar to mechanics.

1. INTRODUCTION

In this paper I shall describe a class of dynamical structures that give a natural extension of electrodynamics to sections of a fibered manifold pair (N, S, π) satisfying $\dim S = n$ and $\dim N = \binom{n}{k} + n$. The construction is based on a representation of the infinitesimal behavior of sections of (N, S, π) in terms of a class of "symplectic" $(k+1)$-vector fields on N that generalize the concept of a cosymplectic structure on a Poisson manifold. Higher degree symplectic structures play a role in the study of multidimensional variational problems that is similar to the role played by symplectic structures in mechanics in that they are exterior derivatives of prolongations of volume forms; see Kijowski and Szczyrba (1975) and Goldschmidt and Sternberg (1973). The present application is based on a separate property of these objects. It shall be shown that the class of "symplectic" $(k+1)$-vector fields function as tangent vectors to sections of (N, S, π). Dynamical structures that are formally analogous to mechanics are constructed from these objects, and yet the resulting dynamical equations are natural generalizations of Maxwell's equations. In fact, when $k = 2$, Maxwell's equations are the dynamical equations for a theory that is formally analogous to Newtonian mechanics.

[1]Department of Mathematics, University of California, Davis, California 95616.

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The remainder of this paper is divided into four sections. Section 2 develops the necessary multilinear algebra of symplectic $(k+1)$-forms. Section 3 uses these results to introduce the geometric structures on which the construction is based. Section 4 gives the dynamical condition and studies the formal properties that establish the relation between this construction and mechanics. In Section 5, a class of structures analogous to Newtonian mechanics is introduced. The resulting dynamical equations are shown to yield Maxwell's equations when $k=2$, and to have traveling wave solutions when $k=3$.

2. SYMPLECTIC $(k+1)$-FORMS

This section develops the linear algebra of the geometric objects that will be used to describe the infinitesimal behavior of sections of a fibered manifold pair (N, S, π). Fix the following notation. If V is a vector space and W is a subspace of V^*, and let ann(W) be the subspace of V annihilated by W. Denote the k-forms on V by $\Lambda_k(V)$ and the k-vectors on V by $\Lambda^k(V)$. Adopt the convention that if A: $V \to V$ is an endomorphism, then the induced map on $\Lambda^k(V)$ is also denoted by A: $\Lambda^k(V) \to \Lambda^k(V)$. An element $\omega \in \Lambda_k(V)$ is *nondegenerate* if, for $u \in V$, $i(u)\omega = 0$ implies $u=0$. If S is an order set, let $\mathscr{A}_k(S)$ be the ordered subsets of S containing k elements. If $S=\{1, \ldots, n\}$, then $\mathscr{A}_k(S) = \mathscr{A}^n_k$. To ease the manipulation of multi-indices, adopt the following conventions. If $\{e_1, \ldots, e_n\}$ is a basis for V and $\alpha \in \mathscr{A}^n_k$, denote $e_{\alpha_1} \wedge, \ldots, \wedge e_{\alpha_k}$ by $e^\wedge_\alpha$. If $\alpha = (\alpha_1, \ldots, \alpha_k)$, let $\alpha_{\hat{i}} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_k)$ and let $(\alpha : j) = (\alpha_1, \ldots, \alpha_k, j)$ if $j \notin \alpha$ and $(\alpha : j) = \varnothing$ otherwise.

Definition 2.1. A nondegenerate $(k+1)$-form $\omega \in \Lambda_{k+1}(V)$ is *symplectic* if there exists a subspace $W \subset V$ such that (i) for all $u, v \in W$, $i(u \wedge w)\omega = 0$, (ii) dim $W = \dim \Lambda_k(V/W)$ and dim $V/W > k$.

Denote the variety of symplectic $(k+1)$-forms in $\Lambda_{k+1}(V)$ by $\mathrm{Sym}_{k+1}(V)$. Clearly, in the case where $k=1$, Definition 2.1 reduces to the definition of a symplectic form on an even-dimensional vector space. In this section we shall only consider the case $k>1$. The following propositions are direct consequences of Definition 2.1.

Proposition 2.1. If $W \subset V$ is a subspace satisfying Definition 2.1, then the map i: $W \to \Lambda_k(V/W)$ given by $i(v) = i(v)\omega$ is an isomorphism.

Proposition 2.2. If $W \subset V$ is a subspace satisfying Definition 2.1, and if $u, v \in W$ with $u \wedge v \neq 0$ and $i(u \wedge v)\omega = 0$, then span$(u, v) \cap W \neq 0$.

Proof. If $\text{span}(u, v) \cap W = 0$, choose $v_1, \ldots, v_{k-2} \in V$ with $v_i \notin W$ such that $u \wedge v \wedge v_1 \wedge \cdots \wedge v_{k-2} \neq 0$ and $\text{span}(u, v, v_1, \ldots, v_{k-2}) \cap W = 0$. But for any $w \in W$, $\omega(w, u, v, v_1, \ldots, v_{k-2}) = 0$, so $\ker(i) \neq 0$. ■

Proposition 2.3. If W' and W are subspaces satisfying Definition 2.1, then $W = W'$.

Proof. By Proposition 2.2, $W' \cap W$ is a subspace of W of at most codimension 1. If $W' \neq W$ then there is $v \in W'$ so that $v \notin W$. Consider the subspace $Z \subset \Lambda^k(V/W)$ given by $Z = \pi(v) \wedge \Lambda^{k-1}(V/W)$, where π is the projection onto V/W. Now, $\dim Z > 1$ and for any $z \in Z$ and any $w \in W' \cap W$, $i(w)(z) = 0$. Therefore, $\ker(i) \neq 0$. ■

Proposition 2.3 states that if $k > 1$, a symplectic $(k+1)$-form uniquely determines the subspace W. For $\omega \in \text{Sym}_{k+1}(V)$ denote this subspace by W_ω. This fact is central to the construction of the following sections. Also, we shall require a canonical representation of symplectic $(k+1)$-forms. The following propositions show that all such forms possess a set of "diagonal" coordinates. To see this, we extend the definition of a Lagrangian subspace to $(k+1)$-forms.

Definition 2.2. A subspace $U \subset V$ is a *Lagrangian subspace* for $\omega \in \Lambda_{k+1}(V)$ if $\omega|_U = 0$ and if U is maximal in the lattice of subspaces with this property.

Note that if $k > 1$, Lagrangian subspaces need not have equal dimension. Also note that if $\omega \in \text{Sym}_{k+1}(V)$ and $k > 1$, then W_ω is not a Lagrangian subspace.

Proposition 2.4. For every $\omega \in \text{Sym}_{k+1}(V)$ there exists a Lagrangian subspace U transverse to W_ω such that $V = U \oplus W_\omega$.

Proof. Suppose that U' is transverse to W_ω and $\omega|_U = 0$, but $U' \oplus W_\omega \neq V$. Let $\dim V/W = n$ and let $\dim U' = l$. Let $\{u_1, \ldots, u_l\}$ be a basis for U' and let $\{u_\alpha^\wedge\}_{\alpha \in \mathcal{A}_k^l}$ be a basis of $\Lambda^k(U')$. Definition 2.1 implies that the set of linear equations $\mathcal{B} = \{\lambda_\alpha \in \Lambda^1(V) | \lambda_\alpha = i(u_\alpha^\wedge)\omega\}$ is independent of W. Consquently, if S is the solution space of $\mathcal{B}$, then

$$\dim S \cap W_\omega = \binom{n}{k} - \binom{l}{k}$$

But,

$$\dim S \geq \binom{n}{k} + n - \binom{l}{k}$$

and so there is $U'' \subset S$ such that $U'' \cap W_\omega = 0$ and $U' \subset U''$. ■

Proposition 2.5. For each $\omega \in \text{Sym}_{k+1}(V)$ there is a basis

$$\{f_\alpha, e_i\}_{\alpha \in \mathcal{A}_k^n i \in \{1, \ldots, n\}}$$

of V such that $\{f_\alpha\}_{\alpha\in\mathscr{A}_k^n}$ is a basis of W_ω and relative to the dual basis

$$\{f_\alpha^*, e_i^*\}_{\alpha\in\mathscr{A}_k^n i\in\{1,\dots,n\}}$$

we have

$$\omega = \sum_{\alpha\in\mathscr{A}_k^n} f_\alpha^* \wedge e_\alpha^{*\wedge} \tag{1}$$

Proof. Let U be a Lagrangian subspace transverse to W_ω and let $\{e_1,\dots,e_n\}$ be a basis for U. If $e_1^*,\dots,e_n^*\in V^*$ form a dual basis to $\{e_1,\dots,e_n\}$ and $e_{j|W_\omega}^* = 0$, then for each $\alpha\in\mathscr{A}_k^n$, define $f_\alpha\in W_\omega$ by $i(f_\alpha)\omega = e_\alpha^{*\wedge}$. The $\{f_\alpha\}_{\alpha\in\mathscr{A}_k^n}$ is a basis for W_ω. Let $\{f_\alpha^*\}_{\alpha\in\mathscr{A}_k^n}$ be the dual basis extended to V, so that $f_{\alpha|U}^* = 0$, and let ω' be the right-hand side of (1). To show that $\omega=\omega'$, just verify that for $u_1,\dots,u_k\in U$ and $w\in W_\omega$, $\omega(w, u_1,\dots,u_k) = \omega'(w,u_1,\dots,u_k)$. ■

There is a cell of Lagrangian subspaces transverse to W_ω. In fact, if U is one such subspace, then in terms of the basis of Proposition 2.5 the others have graphs relative to the splitting $W_\omega\oplus U$ that satisfy for $\alpha\in\mathscr{A}_{k+1}^n$,

$$\sum_j (-1)^j A_{\alpha_{\hat{j}},\alpha_j} = 0,$$

where $A_{\alpha,k}$ are the components of an endomorphism $A\colon U\to W_\omega$ determined by the graph of a Lagrangian subspace transverse to W_ω. In the next section we shall consider symplectic $(k+1)$-forms on manifolds. It is natural to ask whether the coordinates of Proposition 2.5 can be extended C^∞-locally. In the case of differential forms the answer is yes, if the distribution determined by the subspaces W_ω is integrable and if ω is closed. In the case of $(k+1)$-vector fields, the integrability conditions are not understood. For the purposes of the present discussion these results are not required; the existence of a local frame field that is pointwise given by Proposition 2.5 will be sufficient.

3. KINEMATIC CONDITIONS

If N is a smooth manifold and (E, N, π) is a vector bundle over N, denote the bundle of k-forms on the fibers of E by $\Lambda_k(E)$, and denote the antisymmetric degree-k tensor product of E by $\Lambda^k(E)$. Also denote the symmetric degree-k tensor product of E by $S^k(E)$. Denote the space of sections of $\Lambda^k(E)$ by $\mathscr{E}^k(E)$ and the space of sections of $\Lambda_k(E)$ by $\mathscr{E}_k(E)$. Sections of $\Lambda^k(TN)$ are called *k-vector fields.* Now transfer in the standard manner the linear algebraic structures developed in Section 2 to N. The variance of the resulting geometric structures will depend on whether the vector space V in Definition 2.1 is a model for the fibers of TN or T^*N.

If V is a fiber of TN, a *presymplectic k-form* satisfies Definition 2.1 pointwise. In the case where V is a fiber of T^*N, *a presymplectic k-vector field* satisfies Definition 2.1 pointwise. Denote the set of presymplectic k-forms by $\mathrm{Psym}_k(N)$ and the set of presymplectic k-vector fields by $\mathrm{Psym}^k(N)$.

Now we describe how elements of $\mathrm{Psym}^{k+1}(N)$ can be used to specify sections of a fibered manifold pair (N, S, π) satisfying $\dim N = \binom{n}{k} + n$ and $\dim S = n$. Let $\Lambda \in \mathrm{Psym}^{k+1}(N)$ and let W_Λ be a subbundle of T^*N that pointwise satisfies Definition 2.1. Then $Y = \mathrm{ann}(W_\Lambda)$ is an n-dimensional distribution on N. If Y is transverse to the fibers of π and integrates to a foliation $\mathcal{Y}$, then the leaf of $\mathcal{Y}$ through $p \in N$, $\mathcal{Y}_p$, determines a section s of (N, S, π) given by $s = (\pi | \mathcal{Y}_p)^{-1}$. One can associate Λ with s in the same manner that a vector field is associated with an unparametrized integral curve of its flow. We say that Λ is *tangent* to s.

If the dynamics of sections of (N, S, π) is to be formulated in terms of presymplectic $(k+1)$-vector fields, a condition is required that will guarantee that $\Lambda \in \mathrm{Psym}^{k+1}(N)$ is tangent to at least one section of (N, S, π). Such a condition can be given in terms of the Schouten bracket. The Schouten bracket is obtained as an extension of the Lie bracket to contravariant tensors. The extension of the Lie bracket to k-vector fields used here differs from the definition given by Nijenhuis (1955). If $\Lambda \in \mathscr{E}^k(TN)$ and $\Sigma \in \mathscr{E}^l(TN)$, denote the Schouten bracket of Λ and Σ by $[\Lambda, \Sigma] \in \mathscr{E}^{l+k-1}(TN)$. The properties of the new extension are discussed in the Appendix. The integrability condition that implies the existence of a foliation $\mathcal{Y}$ can be stated in terms of the following bundle maps. Given $\Lambda \in \mathrm{Psym}^{k+1}(N)$, for k odd, define, for λ, μ, $\nu \in \mathscr{E}^1(W_\Lambda)$, $g_\Lambda\colon \Lambda^3(W_\Lambda) \to \Lambda^{2k-2}(TN)$ by

$$g_\Lambda(\lambda, \mu, \nu) = i(\lambda \wedge \mu \wedge \nu)[\Lambda, \Lambda].$$

For k even, define, for λ, μ, $\nu \in \mathscr{E}^1(W_\Lambda)$, $g_\Lambda\colon S^3(W_\Lambda) \to \Lambda^{2k-2}(TN)$ by

$$g_\Lambda(\lambda, \mu, \nu) = i(\lambda)[i(\mu)\Lambda, i(\nu)\Lambda] + i(\nu)[i(\lambda)\Lambda, i(\mu)\Lambda] + i(\mu)[i(\nu)\Lambda, i(\lambda)\Lambda].$$

The definition of g_Λ has two parts because the symmetry of the bracket depends upon the degree of the forms; see Definition A.1. Also, the fact that g_Λ is tensorial when k is even follows from Definition 2.1 and the identity for Σ, $M \in \mathscr{E}^{k+1}(TN)$ and $f \in \mathscr{F}(N)$

$$[f\Sigma, M] = -\Sigma \wedge (i(df)M) + f[\Sigma, M].$$

Definition 3.1. $\Lambda \in \mathrm{Psym}^{k+1}(N)$ satisfies the *kinematic Maxwell condition* if $g_\Lambda = 0$.

It shall be shown that under certain conditions the kinematic Maxwell condition guarantees the existence of sections tangent to Λ. The following example justifies the choice of terminology.

Example 3.1. Let $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ be the standard coordinate vector fields on $T^*\mathbb{R}^n$ and let

$$\Lambda = \sum_i P_i \wedge Q_i - \sum_{i<j} E_{ij} P_i \wedge P_j$$

where $E_{ij} \in \mathscr{F}(\mathbb{R}^n)$ with $E_{ij} = -E_{ji}$. If $(dq_1, \ldots, dq_n, dp_1, \ldots, dp_n)$ is the dual basis, then a choice for W_Λ is

$$W_\Lambda = \text{span}(\{dp_i - \tfrac{1}{2}\sum_j E_{ij}\, dq_j\}_{i\in(1,\ldots,n)})$$

A calculation shows that $g_\Lambda = 0$ implies $\nabla_{(k}E_{ij)} = 0$. Thus, Definition 3.1 leads to the kinematic Maxwell equation. The significance of the choice Λ will be discussed in Section 5.

If $k > 1$, the uniqueness of W_Λ implies that any condition that guarantees the existence of a section with tangent Λ must imply the integrability of W_Λ. With a dimension condition, this is true for the kinematic Maxwell condition.

Proposition 3.1. If $k > 2$ and $\dim S > 2k$, then $g_\Lambda = 0$ implies that W_Λ is integrable.

Proof. Assume that k is odd. Let

$$\{P_\alpha, Q_j\}_{\alpha\in\mathscr{A}_k^n j\in(1,\ldots,n)}$$

be a canonical set of basis vector fields for Λ, and let

$$\{p_\alpha^*, q_j^*\}_{\alpha\in\mathscr{A}_k^n j\in(1,\ldots,n)}$$

be the corresponding dual basis of forms. Then

$$\text{ann}(W_\Lambda) = \text{span}(\{Q_j\}_{j\in(1,\ldots,n)}).$$

W_Λ is integrable if, for any $\alpha \in \mathscr{A}_k^n$ and $i, j \in (1, \ldots, n)$, $p_\alpha^*([Q_i, Q_j]) = 0$. If $j \notin \alpha$, choose $l, k \in \alpha - \{i\}$, or if $i \notin \alpha$, choose $l, k \in \alpha - \{j\}$. If $\{i, j\} \subset \alpha$, choose $k \in \alpha - \{i, j\}$ and let $l = i$. Choose λ, $\eta \in \mathscr{A}_{k-2}^n$, so that $\eta \cap \lambda = \varnothing$ and i, j, k, $l \notin \lambda \cup \eta$. Let $\beta = \eta \cup \{i, j\}$ and $\gamma = \lambda \cup \{i, j\}$. Note that

$$[\Lambda, \Lambda] = \sum_{\alpha,\beta} P_\alpha \wedge P_\beta \wedge [Q_\alpha^\wedge, Q_\beta^\wedge] + \Psi$$

where Ψ is a sum of monomials containing at most one P_κ, and so

$$i(p_\alpha^* \wedge p_\beta^* \wedge p_\gamma^*)[\Lambda, \Lambda] = 2(i(p_\alpha^*)[Q_\beta^\wedge, Q_\gamma^\wedge] + i(p_\beta^*)[Q_\gamma^\wedge, Q_\alpha^\wedge] + i(p_\gamma^*)[Q_\alpha^\wedge, Q_\beta^\wedge])$$

But,

$$[Q_\alpha^\wedge, Q_\beta^\wedge] = \sum_{i,j} (-1)^{i+j} [Q_{\alpha_i}, Q_{\beta_j}] \wedge Q_{\hat{\alpha_i}}^\wedge \wedge Q_{\hat{\beta_j}}^\wedge$$

Hence

$$i(p_\alpha^* \wedge p_\beta^* \wedge p_\gamma^*)[\Lambda, \Lambda] = p_\alpha^*([Q_i, Q_j])Q_\eta^\wedge \wedge Q_j \wedge Q_\lambda^\wedge \wedge Q_i + \Delta$$

where Δ is independent of $Q_\eta^\wedge \wedge Q_j \wedge Q_\lambda^\wedge \wedge Q_i$. Thus, $g_\Lambda = 0$ implies $p_\alpha^*([Q_i, Q_j]) = 0$. A similar argument works when k is even. ■

A different argument shows that the conclusion is also true for $k=2$ and $\dim S \geq 4$. It appears that if $\dim S = 2k-1$, $g_\Lambda = 0$ does not imply the integrability of W_Λ. For $k=1$, the Darboux Theorem gives the existence of sections with tangent Λ.

4. DYNAMICAL CONDITIONS

This section describes how an extension of Hamiltonian mechanics induces a dynamical structure on $\mathrm{Psym}^k(N)$. Let Σ be a k-vector field on $\Lambda_k(TN)$. If $\pi: \Lambda_k(TN) \to N$ is the bundle map, Σ determines a map $l: \Lambda^k(TN) \to \Lambda_k(TN)$ given by $l = \pi_* \circ \Sigma$. The situation can be summed up by the following diagram:

$$\begin{array}{ccccc} & & \Lambda^k(T\Lambda_k(TN)) & & \\ & \swarrow^{\pi_*} & & \nwarrow^{\Sigma} & \\ \Lambda^k(TN) & & \xleftarrow{\quad l \quad} & & \Lambda_k(TN) \\ & \searrow & & \swarrow_{\pi} & \\ & & N & & \end{array}$$

When $k=1$, then Σ is a vector field on T^*N, and l is a Legendre transformation. In general we shall refer to l and the *Legendre transformation induced by* Σ. In the following it shall be assumed that l is a diffeomorphism.

$\Lambda_k(TN)$ carries a canonical presymplectic $(k+1)$-form $\Omega \in \mathrm{Psym}_{k+1}(\Lambda_k(TN))$ which is defined by $\Omega = d\sigma$, where σ is the canonical k-form given by $\sigma(p) = \pi^* p$ for $p \in \Lambda_k(TN)$. Note that W_Ω is the vertical bundle $V\Lambda_k(TN)$ over $\Lambda_k(TN)$. If s is a section of $(\Lambda_k(TN), N, \pi)$, then s induces a projection P_s onto the tangent space of its image given by $P_s = s_* \circ \pi_*$. Note that $\ker(P_s) = V\Lambda_k(TN)|_{s(N)}$. The dynamical condition can be stated in terms of Ω, l, and P_s as follows.

Definition 4.1. $\Lambda \in \mathrm{Psym}^k(N)$ satisfies the *dynamical Maxwell condition determined by* Σ if, for $p \in N$,

$$i(P_{l^{-1}(\Lambda)}\Sigma(l^{-1}(\Lambda)(p)) - \Sigma(l^{-1}(\Lambda)(p)))\Omega = 0$$

Note that for any Λ,

$$\pi_*(P_{l^{-1}(\Lambda)}\Sigma(l^{-1}(\Lambda)(p)) - \Sigma(l^{-1}(\Lambda)(p))) = 0$$

The 1-form

$$f_\Lambda = l^{-1}(\Lambda)^* i(P_{l^{-1}(\Lambda)}\Sigma(l^{-1}(\Lambda)(p)) - \Sigma(l^{-1}(\Lambda)(p)))\Omega$$

on N is called the *force on* Λ *determined by* Σ. The f_Λ measures the extent to which Λ satisfies Definition 4.1. The following example demonstrates the connection between Defintion 4.1 and mechanics.

Example 4.1. If Z is a vector field on T^*N and β is the corresponding Legendre transformation, then Z determines a spray $S = \beta_* Z$ on TN. The spray S defines a second-order system of ordinary differential equations on N. If X is a vector field on N, then the integral curves of X are solutions to S if, for $q \in N$, $S(X(q)) = X_* X(q)$. This implies that $Z(\beta^{-1}X(q)) = (\beta^{-1}X)_* X(q)$. Since $\pi_* Z\beta^{-1}X = X$ and since the canonical 2-form is nondegenerate, this condition is equivalent to Definition 4.1.

Example 4.1 suggests that Definition 4.1 should require for $k > 1$ that $\Sigma(l^{-1}(\Lambda)) = l^{-1}(\Lambda)_* \Lambda$. This is much too strong, since, for $q \in N$, $l^{-1}(\Lambda)_* \Lambda(q)$ is generated by subspace $l^{-1}(\Lambda)_* TN_q$ and Σ is in general nondegenerate. Definition 4.1 is the strongest condition that is satisfied for arbitrary Σ.

As in the mechanical example, k-vector fields on $\Lambda_k(N)$ that are generated by functions play a special role. The $\Sigma \in \mathscr{E}^k(\Lambda_k(TN))$ is a Hamiltonian k-vector field if $di(\Sigma)\Omega = 0$. Again, when $k > 1$ the situation differs from the mechanical example in that there is no one-to-one correspondence between exact 1-forms and Hamiltonian k-vector fields. However, if $i(\Sigma)\Omega = dh$, the inverse of l can still be computed as the vertical derivative of the function $\mathrm{l} \in \mathscr{F}(\Lambda^k(N))$ given by $\mathrm{l}(p) = l^{-1}(p)(p) - h(l^{-1}(p))$.

Of particular interest to the present discussion are k-vector fields that satisfy a homogeneity condition. On $\Lambda_k(TN)$ there exists a vertical vector field X_σ defined by $i(X_\sigma)\Omega = \sigma$ that serves as a homogeneity operator on $\Lambda_k(TN)$. Note that equations of this type are not generally solvable; however, in this case a unique solution is guaranteed by Proposition 2.1. X_σ has the property that $L_{X_\sigma}\Omega = \Omega$. The following definition introduces a class of homogeneous k-vector fields for which the dynamical Maxwell condition has a standard form on N.

Definition 4.2. A k-vector field $\Sigma \in \mathscr{E}^k(\Lambda_k(N))$ is *projectable* if (i) there is $c \in \mathbb{R}$ so that $L_{X\sigma}\Sigma = c\Sigma$, and (ii) $i(X_\sigma)di(\Sigma) = 0$.

Note that homogeneous Hamiltonian k-vector fields are projectable.

Proposition 4.1. If Σ is projectable and if $c \neq -1$, then for any k-vector field Λ on N

$$f_\Lambda = i(\Lambda)\, dl^{-1}(\Lambda) - \frac{(-1)^k}{1+c}\, d(l^{-1}(\Lambda)(\Lambda)) \tag{2}$$

Proof. By Definition 4.2(i),

$$L_{X_\sigma} i(\Sigma)\Omega = i(L_{X_\sigma}\Sigma)\Omega + i(\Sigma)L_{X_\sigma}\Omega = (1+c)i(\Sigma)\Omega.$$

But,

$$L_{X_\sigma} i(\Sigma)\Omega = di(X_\sigma)i(\Sigma)\Omega + i(X_\sigma)\, di(\Sigma)\Omega$$

and so Definition 4.2(ii) implies $(-1)^k\, d\sigma(\Sigma) = (1+c)i(\Sigma)\Omega$. Thus

$$f_\Lambda = l^{-1}(\Lambda)^* i(l^{-1}(\Lambda)_* \Lambda)\Omega - l^{-1}(\Lambda)^* i(\Sigma)\Omega$$

$$= i(\Lambda) l^{-1}(\Lambda)^*\Omega - \frac{(-1)^k}{1+c}\, l^{-1}(\Lambda)^*\, d\sigma(\Lambda)$$

and since $l^{-1}(\Lambda)^*\sigma(\Sigma) = l^{-1}(\Lambda)(\Lambda)$ and $l^{-1}(\Lambda)^*\Omega = dl^{-1}(\Lambda)$, equation (2) follows. ∎

Equation (2) is a generalization to vector fields of arbitrary degree of the intrinsic expression for the covariant derivative of a vector field relative to itself with respect to Levi-Civita connection. It leads to an extension of Riemannian geometry to k-vector fields that is beyond the scope of the present discussion.

5. NEWTONIAN MODELS

To see that for $\Lambda \in \mathrm{Psym}^{k+1}(N)$ the equations $g_\Lambda = 0$ and $f_\Lambda = 0$ are an extension of the free field Maxwell equations, we shall consider a particular model of the previous construction. Let $N = \Lambda_k(TS)$ and let g be a pseudo-Riemannian metric on S. The Levi-Civita connection on S induces a connection ∇ on $\Lambda_k(TS)$. The connection ∇ determines a horizontal distribution H on $\Lambda_k(TS)$ so that $T\Lambda_k(TS) = V\Lambda_k(TS) \oplus H$. Let P: $T\Lambda_k(TS) \to H$ be the bundle map determined by projection onto the horizontal. For horizontal vector fields U and V, $R(U, V) = (1-P)[U, V]$ is the $(2, 1)$-tensor that determines the curvature of ∇. Extend g to a metric g' on $\Lambda_k(TS)$ so that g' is the orthogonal direct sum of the lifted metric on H and the fiber metric on $V\Lambda_k(S)$. In turn g' induces a fiber metric on $\Lambda_{k+1}(TN)$. Define a Hamiltonian function on $\Lambda_{k+1}(TN)$ by $h(p) = g'(p, p)$ for $p \in \Lambda_{k+1}(N)$. Let Σ be a homogeneous Hamiltonian $(k+1)$-vector field on $\Lambda_{k+1}(TN)$ determined by h. We seek solutions to the dynamical Maxwell condition that are presymplectic and also satisfy the following definition. Let Ω be the canonical $(k+1)$-form on $\Lambda_k(TN)$.

Definition 5.1. $\Lambda \in \mathscr{E}^{k+1}(TN)$ is *Newtonian* if (i) $\pi_*\Lambda = 0$ and (ii) for all $\lambda \in \mathscr{E}^1(\mathrm{ann}(H))$, $\pi_*(i(\lambda)(l(\Omega) - \Lambda)) = 0$.

The following proposition helps explain the choice of terminology.

Proposition 5.1. If $\Lambda \in \text{Psym}^{k+1}(N)$ is Newtonian, then, for $\lambda_1, \ldots, \lambda_{k+1} \in \mathscr{E}_1(TN)$,

$$\Lambda(\lambda_1, \ldots, \lambda_{k+1}) = \sum_i l(\Omega)((P+A)^t\lambda_1, \ldots, \lambda_i, \ldots, (P+A)^t\lambda_{k+1}) \tag{3}$$

where $A: H \to V\Lambda_k(TS)$ is the bundle map with graph $\text{ann}(W_\Lambda)$.

Proof. First note that Definition 5.1(i) implies that $\text{ann}(W_\Lambda)$ is transverse to $V\Lambda_k(TS)$ and so A is well defined. Now, $\text{ann}(W_{l(\Omega)}) = H$. For $p \in N$, let

$$\{p_\alpha^*, q_j^*\}_{\alpha \in \mathscr{A}_k^n j \in (1,\ldots,n)}$$

be a basis of T^*N_p dual to a canonical basis for $l(\Omega)(p)$ such that $\{p_\alpha^*\}_{\alpha \in \mathscr{A}_k^n}$ spans $W_{l(\Omega)}$. Then $p_\alpha'^* = p_\alpha^* - \sum_j A_{\alpha,j} q_j^*$ is a basis for W_Λ. If Λ' is the right-hand side of (3), to show that $\Lambda = \Lambda'$, one must verify that $\Lambda'(p_\alpha'^*, q_\beta^{*\wedge}) = \Lambda(p_\alpha'^*, q_\beta^{*\wedge})$. But, $\Lambda'(p_\alpha'^*, q_\beta^{*\wedge}) = l(\Omega)(p_\alpha^*, q_\beta^{*\wedge})$ and

$$i(p_\alpha'^*)(l(\Omega) - \Lambda)(q_\beta^{*\wedge}) = 0 \qquad ■$$

Therefore, if $\Lambda \in \text{Psym}^{k+1}(N)$ is Newtonian, Λ is completely determined once W_Λ is known. This property is the analog of the fact that in Newtonian mechanics the velocity completely determines the tangents to world lines. $A: H \to V\Lambda_k(TS)$ can be thought of as the velocity of Λ.

To find the local expression for (2) when Λ is Newtonian, introduce an orthonormal frame field $\{Q_1', \ldots, Q_n'\}$ on S, and let $\{q_1'^*, \ldots, q_n'^*\}$ be the dual basis of 1-forms. For $\alpha \in \mathscr{A}_k^n$, define $p_\alpha'^* \in \mathscr{E}_k(TS)$ by $p_\alpha'^* = q_\alpha'^{*\wedge}$. Extend $p_\alpha'^*$ by affine translation to a vector field P_α along $V\Lambda_k(TS)$, and let Q_j be the horizontal lift of Q_j'. Then

$$\{P_\alpha, Q_j\}_{\alpha \in \mathscr{A}_k^n j \in (1,\ldots,n)}$$

is an orthonormal frame field for $\Lambda_k(TS)$ and $\Omega(P_\alpha, Q_\beta^\wedge) = \delta_{\alpha\beta}$. Let

$$\{p_\alpha^*, q_j^*\}_{\alpha \in \mathscr{A}_k^n j \in (1,\ldots,n)}$$

be the dual fields. If $Q_j'^*$ is the metric dual of Q_j', then there is $\varepsilon_j = \pm 1$, so that $q_j'^* = \varepsilon_j Q_j'^*$. For $\alpha \in \mathscr{A}_k^n$ let $\varepsilon_\alpha = \varepsilon_{\alpha_1} \cdot, \ldots, \cdot \varepsilon_{\alpha_k}$. Now let $\bar{\mathscr{A}}_l = \mathscr{A}_l(\mathscr{A}_k^n)$ and let $\mathscr{A}_k = \mathscr{A}_k^n$. Definition 5.1 implies that a Newtonian $\Lambda \in \mathscr{E}^{k+1}(TN)$ is of the form

$$\Lambda = \sum_\alpha P_\alpha \wedge Q_\alpha^\wedge + \sum_{j>1} \sum_{\substack{\lambda \in \bar{\mathscr{A}}_j \\ \mu \in \mathscr{A}_{k-j+1}}} E_{\lambda_1 \ldots \lambda_j, \mu} P_\lambda^\wedge \wedge Q_\mu^\wedge \tag{4}$$

and also

$$l^{-1}(\Lambda) = \sum_\alpha p_\alpha^* \wedge q_\alpha^{*\wedge} + \sum_{j>1} \sum_{\substack{\lambda \in \bar{\mathscr{A}}_j \\ \mu \in \mathscr{A}_{k-j+1}}} \bar{E}_{\lambda_1 \ldots \lambda_j, \mu} p_\lambda^{*\wedge} \wedge q_\mu^{*\wedge}$$

where $\bar{E}_{\lambda_1\ldots\lambda_j,\mu} = \varepsilon_\lambda \varepsilon_\mu E_{\lambda_1\ldots\lambda_j,\mu}$. In terms of these expressions the dynamical Maxwell condition has the following form on Newtonian $(k+1)$-vector fields.

Proposition 5.2. Let $\Lambda \in \mathscr{E}^{k+1}(TN)$ be Newtonain, and suppose that, for all $\alpha \in \mathscr{A}_k^n$, $L_{P_\alpha}\Lambda = 0$. Then the force on Λ is given by

$$f_\Lambda = \sum_\mu (-1)^{k+1} f'_\mu p_\mu^* - \sum_l f''_l q_l^*$$

where

$$f'_\mu = \sum_{\alpha,i} (-1)^i \nabla_{\alpha_i} \bar{E}_{\alpha\mu,\alpha_i^\wedge} + \sum_{\substack{\kappa,\alpha\\ r<s}} (-1)^{r+s} \bar{E}_{\alpha\mu\kappa,(\alpha_r^\wedge)_s^\wedge} R^\kappa_{\alpha_r\alpha_s}$$
$$+ \sum_{\substack{i,j,\lambda\in\bar{\mathscr{A}}_j\\ \gamma\in\mathscr{A}_{k-j+1}}} (-1)^i E_{\lambda,\gamma} \nabla_{\gamma_i} \bar{E}_{(\lambda:\mu),\gamma_i^\wedge}$$
$$+ \sum_{\substack{j,\lambda\in\bar{\mathscr{A}}_j\\ \gamma\in\mathscr{A}_{k-j+1}}} \sum_{\substack{\kappa\\ r<s}} (-1)^{r+s} E_{\lambda,\gamma} \bar{E}_{(\lambda:\mu\kappa),(\gamma_r^\wedge)_s^\wedge} R^\kappa_{\gamma_r\gamma_s}$$

$$f''_l = \sum_{\substack{\kappa,\alpha\\ r<s}} (-1)^{r+s} \bar{E}_{\alpha\kappa,((\alpha:l)_r^\wedge)_s^\wedge} R^\kappa_{(\alpha:l)_r(\alpha:l)_s} + \sum_{\substack{i,j,\lambda\in\bar{\mathscr{A}}_j\\ \gamma\in\mathscr{A}_{k-j+1}}} (-1)^{i+j} E_{\lambda,\gamma} \nabla_{\gamma_i} \bar{E}_{\lambda,(\gamma_i^\wedge,l)}$$
$$+ \sum_{\substack{j,\lambda\in\bar{\mathscr{A}}_j\\ \gamma\in\mathscr{A}_{k-j+1}}} E_{\lambda,\gamma} \sum_{\substack{\kappa\\ r<s}} (-1)^{r+s+j} \bar{E}_{(\lambda:\kappa),((\gamma:l)_r^\wedge)_s^\wedge} R^\kappa_{(\gamma:l)_r(\gamma:l)_s}.$$

Proof. Choose $\{Q'_1, \ldots, Q'_n\}$ so that, at $q \in S$, $[Q'_i, Q'_j](q) = 0$ and $\nabla_{Q'_j} Q'_i(q) = 0$ for all i and j. Then, for $x \in N$ with $\pi(x) = q$, $dq_j^*(x) = 0$ and $dp_\alpha^*(x)_{\beta,i} = dp_\alpha^*(x)_{\beta\gamma} = 0$ and $dp_\alpha^*(x)_{ij} = R_{ij}^\alpha(x)$; see Dombrowski (1962). Consequently,

$$dl^{-1}(\Lambda) = \sum_{\substack{j>1,\lambda\in\bar{\mathscr{A}}_j\\ \gamma\in\mathscr{A}_{k-j+1}}} d\bar{E}_{\lambda,\gamma} \wedge p_\lambda^{*\wedge} \wedge q_\gamma^{*\wedge}$$
$$- \sum_{\substack{j>1,\lambda\in\bar{\mathscr{A}}_j\\ \gamma\in\mathscr{A}_{k-j+2}}} (-1)^j \sum_{\substack{\kappa\\ r<s}} (-1)^{r+s} \bar{E}_{(\lambda:\kappa),(\gamma_r^\wedge)_s^\wedge} R^\kappa_{\gamma_r\gamma_s} p_\lambda^{*\wedge} \wedge q_\gamma^{*\wedge}$$

The result follows from (4) and (2). ■

Example 5.1. When $k=1$ and S is Lorentzian, Newtonian 2-vector fields have the form given in Example 3.1. For $k=1$, $\alpha_i^\wedge = \varnothing$ for $\alpha \in \mathscr{A}_1^n$, and so Proposition 5.2 reduce to

$$f_\Lambda = \sum_{i,j} \nabla_j \bar{E}_{ij} p_i^* + \sum_{l<j,i} \bar{E}_{lj} R^l_{ji} q_i^*$$

This can also be written as

$$f_\Lambda = \sum_i (\operatorname{div} \bar{F})_i p_i^* + \tfrac{1}{2} \sum_{l,j,i} \bar{E}_{lj} R^i_{lj} q_i^*$$

where $\bar{F} \in \mathscr{E}_2(TS)$ is the 2-form on S defined by the vertical part of $l^{-1}(\Lambda)$. Thus, the vertical component of f_Λ is the divergence of the field strength, and in flat space the horizontal component vanishes. This is analogous to the situation in Newtonian mechanics, where in flat space the time component of the force vanishes.

Example 5.2. For $k > 1$ the dynamical Maxwell condition leads to nonlinear systems of partial differential equations. This example shows that, although when $k = 2$ Proposition 5.2 gives a nonlinear system, the flat space equations $f_\Lambda = 0$, $g_\Lambda = 0$ still have traveling wave solutions. Since we are assuming S to be flat, let $S = \mathbb{R}^n$. Then $N = \mathbb{R}^n \oplus \Lambda_2(\mathbb{R}^n)$ and the expression for the force is

$$f_\Lambda = \sum_\mu \Big(\sum_\alpha (\nabla_{\alpha_1} \bar{E}_{\mu\alpha,\alpha_2} - \nabla_{\alpha_2} \bar{E}_{\mu\alpha,\alpha_1}) + \sum_{j,\kappa<\lambda} E_{\kappa\lambda,j} \nabla_j \bar{E}_{\mu\kappa\lambda}\Big) p^*_\mu$$
$$+ \sum_l \Big(\sum_{j,\kappa<\lambda} \bar{E}_{\kappa\lambda,j} \nabla_j E_{\kappa\lambda,l}\Big) q^*_l \tag{5}$$

Proposition 5.3. Equation (5) admits presymplectic traveling wave solutions $\Lambda \in \mathrm{Psym}^3(\mathbb{R}^n \oplus \Lambda_2(\mathbb{R}^n))$ that satisfy $g_\Lambda = 0$.

Proof. Using (3), it is easy to see that

$$E_{(ij)(kl),m} = \delta^k_m A_{(ij),l} - \delta^l_m A_{(ij),k} + \delta^j_m A_{(kl),i} - \delta^i_m A_{(kl),j} \tag{6}$$

$$E_{(ij)(kl)(mn)} = (A_{(ij),l} A_{(mn),k} - A_{(ij),k} A_{(mn),l})$$
$$+ (A_{(lk),i} A_{(mn),j} - A_{(lk),j} A_{(mn),i})$$
$$+ (A_{(ij),m} A_{(lk),n} - A_{(ij),n} A_{(lk),m}) \tag{7}$$

Let e_0, e_1, e_2 be orthonormal unit vectors such that on $U = \mathrm{span}(e_0, e_1, e_2)$, g has signature $(-, +, +)$. For $x \in U$, write $x = \sum_{i=0}^2 x_i e_i$. For $(i, j) \in \mathscr{A}_2(\{0, 1, 2\})$ let $f_{ij}: \mathbb{R} \to \mathbb{R}$ be C^2. Define, for $(i, j) \in \mathscr{A}_2(\{0, 1, 2\})$ and $k \in \{0, 1, 2\}$, $A_{(ij),k}(x) = D_k f_{ij}(x_0 - x_1)$, and $A_{(ij),k}(x) = 0$ otherwise. First note that (7) implies that $E_{(01)(02)(12)} = 0$, and so (5) reduces to

$$f_\Lambda = \sum_{(lk)} \left(\sum_{(ij)} \nabla_i \bar{E}_{(ij)(kl),j} - \nabla_j \bar{E}_{(ij)(lk),i}\right) p^*_{(kl)} + \sum_n \left(\sum_{\substack{m \\ (ij)<(kl)}} E_{(ij)(kl),m} \nabla_m \bar{E}_{(ij)(kl),n}\right) q^*_n$$

Now (6) implies that $E_{(01)(12),0} = -A_{(12),1}$, $E_{(01)(12),1} = A_{(12),0}$, $E_{(01)(12),2} = -A_{(01),1}$, $E_{(01)(02),0} = -A_{(02),1}$, $E_{(01)(02),1} = A_{(02),0}$, $E_{(01),(02),2} = -A_{(01),0}$, $E_{(02)(12),0} = E_{(02)(12),1} = 0$, $E_{(02)(12),2} = A_{(12),0} - A_{(02),1}$. Substituting into the above expression for f_Λ shows that $f_\Lambda = 0$ if $(f'_{(12)} + f'_{(02)})' = 0$. The kinematic condition also follows from these identities. ■

Note that the condition $f'_{(12)}+f'_{(02)}=0$ and the transverse wave condition $f_{(01)}=0$ imply that the 2-form $\omega=\sum f_{(ij)}q_i^* \wedge q_j^*$ is a solution to Maxwell's equations. Although it is not true for arbitrary solutions, this shows that traveling wave solutions to the second-order Maxwell condition prolong to solutions of the third-order Maxwell condition.

6. CONCLUDING REMARKS

I conclude with two remarks. First, to summarize, these arguments have shown that in the category of presymplectic $(k+1)$-vector fields, Hamilton's equations on $\Lambda_{k+1}(TN)$ can be interpreted as defining a dynamical structure for sections of the fibered manifold pair (N, S, π). Further, sections of (N, S, π) can be viewed as potentials for an extended electrodynamics. Note that $k=1$ is a degenerate case of this construction, since the arguments of Section 2 do not apply to 2-forms. This degeneracy gives rise to the gauge symmetry of electrodynamics. Gauge symmetries are not present in theories with $k>1$.

The second remark concerns the significance of Newtonian models introduced in Section 5. A Newtonian model can be viewed as an approximation technique that decomposes a motion into forced and free components. For instance, in Example 3.1 the electromagnetic field is represented by the 2-vector field $\Lambda=\sum P_i \wedge Q_i - \sum_{i,j} E_{ij} P_i \wedge P_j$. Now if $h=\frac{1}{2}\sum g_{ij}p_i p_j$ is the mechanical free Hamiltonian on $T^*\mathbb{R}^n$, then the dynamical vector field on $T^*\mathbb{R}^n$ that determines the spray for the Lorentz force law is $i(dh)\Lambda$. Newtonian models for $k>1$ can be given a similar interpretation. In Martin (1987), I describe a non-Newtonian model for electrodynamics. In this model the Newtonian condition is replaced a constant charge condition that is analogous to the condition in relativistic mechanics that requires timelike trajectories to have constant mass. The Newtonian model for $k=2$ describes an "electrodynamics" for this "relativized" version of classical electrodynamics. In this example the fields of force determine the classical electromagnetic currents. The details of the relation between the Newtonian $k=2$ theory and classical electrodynamics shall appear in a subsequent article.

APPENDIX

This Appendix describes an extension of the Lie bracket on vector fields to the algebra of k-vector fields that is similar to the extension given by Nijenhuis (1955), but has a more symmetric derivation property and a simpler local form. If R is a ring and V is an R module, let $\Lambda_R^*(V)$ be the exterior algebra of V over R. So

$$\bigoplus_{k=1}^{n} \mathscr{E}^k(N) = \Lambda^*_{\mathscr{F}(N)}(\mathscr{E}^1(N))$$

Definition A.1. The *Schouten bracket* is that extension of the Lie bracket on $\mathscr{E}^1(N)$ to $\Lambda^*_{\mathscr{F}(N)}\mathscr{E}^1(N)$ that satisfies the following for $\Lambda \in \mathscr{E}^p(N)$, $M \in \mathscr{E}^q(N)$, $N \in \mathscr{E}^r(N)$:

(i) $[\Lambda, M] \in \mathscr{E}^{p+q-1}(N)$,
(ii) $[\Lambda, M] = (-1)^{pq+p+q}[M, \Lambda]$,
(iii) $[\Lambda, M \wedge N] = [\Lambda, M] \wedge N + (-1)^{(p+1)q} M \wedge [\Lambda, N]$.

In principle, Definition A.1 gives an extension of the Lie bracket to $\Lambda^*_{\mathbb{R}}\mathscr{E}^1(N)$. So it remains to prove the bracket factors to $\Lambda^*_{\mathscr{F}(N)}\mathscr{E}^1(N)$. This can be seen by choosing coordinate vector fields $\{X_1, \ldots, X_n\}$ and verifying for all f, $g \in \mathscr{F}(N)$ and any i, j that

$$[X_{\alpha_1} \wedge \cdots \wedge (fX_{\alpha_i}) \wedge \cdots \wedge X_{\alpha_p}, (gX_{\beta_1}) \wedge \cdots \wedge X_{\beta_p}\}]$$
$$= [X_{\alpha_1} \wedge \cdots \wedge (fX_{\alpha_j}) \wedge \cdots \wedge X_{\alpha_p}, (gX_{\beta_1}) \wedge \cdots \wedge X_{\beta_q}]$$

Nijenhuis (1955) uses the condition $[\Lambda, M] = (-1)^{pq}[M, \Lambda]$ instead of (ii). However, Definition A.1 also implies that

$$[M \wedge N, \Lambda] = M \wedge [N, \Lambda] + (-1)^{(p+1)r}[M, \Lambda] \wedge N$$

The bracket given by Definition A.1 satisfies identities similar to Nijenhuis (1955). However, there are differences in sign, and so we list the basic properties in the following propositions.

Proposition A.1. If $\{X_1, \ldots, X_n\}$ is a basis of coordinate vector fields, that is, $[X_i, X_j] = 0$ for all i, j, and if f, $g \in \mathscr{F}(N)$, then for $\alpha \in \mathscr{A}^n_p$ and $\beta \in \mathscr{A}^n_q$

$$[fX_{\hat{\alpha}}, gX_{\hat{\beta}}] = \sum_j (-1)^{p+j}(fX_{\alpha_j}g)X_{\hat{\hat{\alpha_j}}} \wedge X_{\hat{\beta}} + \sum_j (-1)^j (gX_{\beta_j}f)X_{\hat{\alpha}} \wedge X_{\hat{\beta_j}}$$

The Palais formula for the exterior derivative can be formulated in terms of the Schouten bracket.

Proposition A.2. If $\omega \in \mathscr{E}_{p+q-1}(TN)$ and $M \in \mathscr{E}^p(TN)$, $N \in \mathscr{E}^q(TN)$, then

$$(-1)^p\, d\omega(M \wedge N) = d(i(M)\omega)(N) + (-1)^{pq+p+q}\, d(i(N)\omega)(M) + \omega([M, N])$$

Proposition A.2 implies the following useful identity. For μ, $\lambda \in \mathscr{E}_1(TN)$ and for $\Lambda \in \mathscr{E}^p(TN)$ and $M \in \mathscr{E}^q(TN)$, define $\mu \wedge \lambda(\Lambda : M) \in \mathscr{E}^{p+q-1}(TN)$ by

$$\mu \wedge \lambda(\Lambda : M) = i(\mu)\Lambda \wedge i(\lambda)M - i(\lambda)\Lambda \wedge i(\mu)M$$

For $\lambda \in \mathscr{E}_1(TN)$ the Schouten bracket satisfies the derivation rule

$$i(\lambda)[\Lambda, M] = [i(\lambda)\Lambda, M] + (-1)^{p+1}[\Lambda, i(\lambda)M] + (-1)^{p+1}\, d\lambda(\Lambda : M)$$

Finally, Definition A.1 implies the following Jacobi identity.

Proposition A.3. If $\Lambda \in \mathscr{E}^p(TN)$, $M \in \mathscr{E}^q(TN)$, and $N \in \mathscr{E}^r(TN)$, then

$$(-1)^{(q+1)(p+r)}[\Lambda, [M, N]] + (-1)^{(p+1)(r+q)}[N, [\Lambda, M]] + (-1)^{(r+1)(q+p)}[M, [N, \Lambda]] = 0$$

REFERENCES

Dombrowski, P. (1962). On the geometry of the tangent bundle, *J. Reine Angew. Math.*, **210**, 73-88.

Goldschmidt, H., and Sternberg, S. (1973). The Hamilton-Cartan formalism in the calculus of variations, *Ann. Inst. Fourier* (*Grenoble*), **23**, 203-267.

Kijowski, S., and Szczyrba, W. (1975). Multisymplectic manifolds and the geometric construction of the Poisson bracket in field theory, in *Geométrie Symplectique et Physique Mathématique*, pp. 347-378, CNRS, Paris.

Martin, G. (1987). Relation between charge and energy conservation in a nonlinear electrodynamics, *Int. J. Theor. Phys.*, **26**, 873-888.

Nijenhuis, A. (1955). Jacobi-type identities for bilinear differential concomitants of certain tensor fields, *Nederl. Akad. Wetensch. A*, **58**, 390-403.